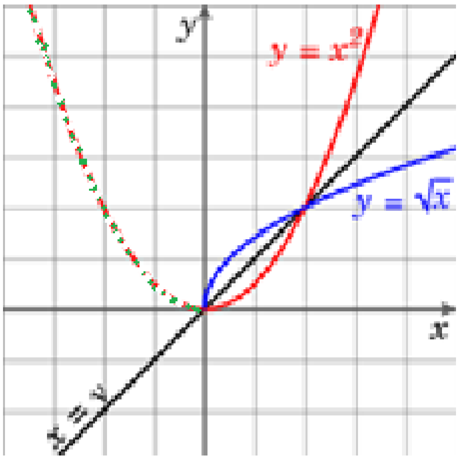
Subsequences revisited.

For a function $f: X \to Y$ and any subset A of X we use the term *restriction* for the function $f_{|A}: A \to Y$ whose domain is A and for each $a \in A$ $f_{|A}(a) = f(a)$.

For example, ${x^2}_{|<0;\infty)}$ becomes a 1-1 function.



The solid red line is the graph of function f(x)=x^2 restricted to <0;∞). The restricted function is 1-1 (the original is not), hence invertible. The blue line is the graph of its inverse, $\sqrt{x}$. The spotted red line is the graph of x^2 restricted to $(-\infty; 0>$. (*Picture from Wikipedia*).

Since a sequence is a function from ℕ into ℝ you may think of a subsequence of a sequence as a restriction of the sequence to an infinite subset of ℕ. („*As a what of what to what*?!” – Winnie the Pooh).

Comprehension (subsequences).

What kind of a sequence is

(a) (a_{2n}) if (a_n) is an arithmetic sequence with the increment *d*,

(b) (a_{2n}) if (a_n) is a geometric sequence with the quotient *q*,

(c) $(a_{2^n})_{n=0}^{\infty}$ if (a_n) is the arithmetic sequence with the increment *d* and $a_1=d$?

Comprehension (geometric sequences).

Assuming daily infection rate of 10% (i.e. every day the number of **new** infections is 10% more than the previous day) and there was a single infected person at the beginning how long does it take to infect 8 billion individuals? Ignore the fact that the model is unrealistic because with the significant fraction of the population infected it becomes harder and harder to find a new individual to infect (also you cannot infect 1.1 people).

Limit of a sequence.

Definition. For every sequence (a_n) the sequence is called *convergent* iff

$(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists k \epsilon \mathbb{N})(\forall m \epsilon \mathbb{N})(k < m \Rightarrow |a_m - L| < \varepsilon)$.

The number L is called *the limit of* a_n *as n approaches* ∞.

We write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.

Otherwise the sequence is called *divergent*.

Notice that the sequence is convergent to L if, whenever somebody chooses an $\varepsilon>0$, you can find such an index k that all terms of the sequence with indices greater than k belong to the interval $(L-\varepsilon; L+\varepsilon)$. This means that for only finitely many terms of (a_n) their distance from L may be $\geq \varepsilon$.

FAQ 1. Does this mean that a sequence is convergent to L iff for every ε infinitely many of its terms are "ε - close" to L?

FAQ 2. Can a sequence have two (or more) limits?

FAQ 3. Can a convergent sequence have two subsequences converging to two different limits?

NO to all three. Suppose L and K are limits to a_n and $K\neq L$. Put

$\varepsilon = \frac{|K-L|}{2}$ i.e. ε is one half of the distance from K to L. Clearly

$$(L-\varepsilon; L+\varepsilon) \cap (K-\varepsilon; K+\varepsilon) = \emptyset$$

so if for some k all terms of the sequence with indices $>k$ belong to $(L-\varepsilon; L+\varepsilon)$ then they do not belong to $(K-\varepsilon; K+\varepsilon)$.

Comprehension 1. Why is this the answer to all three FAQs?

Comprehension 2. Why $(L-\varepsilon; L+\varepsilon) \cap (K-\varepsilon; K+\varepsilon) = \emptyset$?

Theorem 1. A sequence a_n is convergent iff there exists L such that all its subsequences converge to L. If this happens to be the case, then L is also the limit of the sequence a_n.

Proof. ($\Leftarrow$)Obviously (a_n) is its own subsequence, hence if all subsequences converge to L then, in particular (a_n).

($\Longrightarrow$)The same argument we used for FAQ 1,2,3 (slightly modified) works here. (**How?**)

Theorem 1 is commonly used to prove divergence of a sequence:

It is enough to find two subsequences convergent to different limits (e.g. odd- and even-subscripted terms of the sequence $(-1)^n$).

FAQ. Is this the only way to show that a sequence is divergent?

On the contrary, a divergent sequence may contain no convergent subsequences, hence no subsequences convergent to different limits.

Theorem 2. A sequence a_n is convergent iff there exists L such that all its subsequences converge to L. If this happens to be the case, then L is also the limit of the sequence a_n.

Proof. (⇐)Obviously (a_n) is its own subsequence, hence if all subsequences converge to L then, in particular (a_n).

(⟹)The same argument we used for FAQ 1,2,3 (slightly modified) works here.

Comprehension. How do you modify that argument?

Theorem 2 is commonly used to prove divergence of a sequence:

It is enough to find two subsequences convergent to different limits (e.g. odd- and even-subscripted terms of the sequence $(-1)^n$).

Theorem 3. (Arithmetic properties of the limit)

If a_n is convergent to A and b_n to B. Then

(1) a_n+b_n is convergent and $\lim(a_n+b_n) = A + B$,

(2) a_n-b_n is convergent and $\lim(an-bn) = A - B$,

(3) $a_n \cdot b_n$ is convergent and $\lim(a_n \cdot b_n) = AB$,

(4) for every constant $c \in \mathbb{R}$, (ca_n) is convergent and $\lim(ca_n) = cA$

(5) $(\frac{a_n}{b_n})$ is convergent and $\lim(\frac{a_n}{b_n}) = \frac{A}{B}$ (if $b_n \neq 0$ and $B \neq 0$).

In short, arithmetic operations on convergent sequences *preserve* limits, (the limit of the sum is the sum of the limits etc.).

Proof. (Outline of a proof of part 3). The starting point is, as always, the fundamental question, what the hell must we do. We will apply the definition to the sequence $a_n \cdot b_n$

$(\forall\varepsilon > 0)(\exists k\epsilon\mathbb{N})(\forall m\epsilon\mathbb{N})(k < m \Rightarrow |a_m b_m - AB| < \varepsilon)$

$|a_m b_m - AB| = |a_m b_m - a_m B + a_m B - AB| =$

$= |a_m(b_m - B) + (a_m - A)B| \leq$

$\leq |a_m(b_m - B)| + |(a_m - A)B| =$

$= |a_m||b_m - B| + |a_m - A||B|$

Here comes the tricky part – since the sequences converge to A and B we know that choosing sufficiently large m we can make each $|a_m||b_m - B|$ and $|a_m - A||B|$ as small as we like, for example smaller than $\frac{\varepsilon}{2}$, hence the sum will be less than ε. I will spare you the details

Comprehension 2. Notice that Theorem 3 is phrased as an implication (one-way implication) or rather a number of implications. Which (if any) of (1), (2) ,.., (5) are true in the opposite direction?

Theorem 4 (Limits and inequalities)

If a_n is convergent to A and b_n to B and there exists k such that for every $n>k$ $a_n \leq b_n$ then $A \leq B$. (The order is preserved by *lim*).

Outline of a proof. Proof by contradiction. If A>B then we put

$2\varepsilon = A-B$. There is k such that for every $n>k$ a_n must be near A, and b_n near B, '*near*' meaning at a distance smaller than $\varepsilon = \frac{A-B}{2}$. In particular this means $a_n > A - \frac{A-B}{2} = \frac{A+B}{2}$ and $b_n < B + \frac{A-B}{2} = \frac{A+B}{2}$, hence for all $n>k$ we have $a_n > b_n$ – contrary to our assumption.

Theorem 5 (Sandwich theorem, squeeze lemma)

If a_n and b_n are both convergent to L and there exists k such that for every $n>k$ $a_n \leq c_n \leq b_n$ then the sequence c_n is also convergent and also to L.

Outline of a proof. Roughly speaking, if we want to guarantee that terms c_n are closer to L than ε it is enough to choose indices so large, that both a_n and b_n are ε-close to L. This guarantees that $L-\varepsilon < a_n \leq c_n \leq b_n < L+\varepsilon$.

This theorem is incredibly useful. There are tons of sequences whose limits cannot be calculated by arithmetic operations on known, elementary limits and with the squeeze theorem they become … trivial. For example, consider

$a_n = \frac{\sin n}{n}$.

$\lim\limits_{n\to\infty} \sin n$ does not exist (which is not quite trivial) but:

for each n $\quad \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

$\lim\limits_{n\to\infty} \frac{-1}{n} = 0$ and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

Hence, by squeeze lemma, $\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0$.

Common pitfalls.

A student is instructed to check if a sequence x_n is convergent. He recalls "Uncle Tom said something about hamburgers". He finds some sequences z_n and y_n. Now it may go several ways

- z_n and y_n converge to the same limit so he announces proudly that x_n is convergent. But he never bothers to check if $(\forall n)\ z_n \le x_n \le y_n$. LOL, score 0.
- $(\forall n)\ z_n \le x_n \le y_n$ but y_n or z_n is divergent. Whatever his conclusion it makes no sense, LOL.
- $(\forall n)\ z_n \le x_n \le y_n$, z_n and y_n converge but to different limits. Just as the last one.

  I've seen those many times and I don't want to see them again. Or else …

Theorem 6.

Every convergent sequence is bounded.

Comprehension. Prove it from elementary principles (it means directly from the definition of the limit).

Theorem 7.

Not every bounded sequence is convergent, but every bounded **and monotonic** sequence is.

Comprehension. Find a divergent and bounded sequence.

We skip the proof of the second part. It involves the concept of the *least-upper-bound* of a bounded set of real numbers which we have not discussed (we will, in the context of functions).

Theorem 8.

The sequence $(1+\frac{1}{n})^n$ is convergent.

Hint. It turns out that the sequence is increasing and bounded from above, hence convergent by Theorem 5.

For those interested – a detailed proof can be found in the slide show we were using for ETMAG lectures in the good old days.

Definition

We say that a sequence a_n *diverges to* ∞ iff

$(\forall r \in \mathbb{R})(\exists k \epsilon \mathbb{N})(\forall n > k)\ a_n > r$

We denote this by $\lim_{n\to\infty} a_n = \infty$

In a similar way we define *divergence to* $-\infty$:

$(\forall r \in \mathbb{R})(\exists k \epsilon \mathbb{N})(\forall n > k)\ a_n < r$.

So, in total, a sequence may be *convergent* (to a number), *divergent* or *divergent to* (plus or minus infinity). Note that $+\infty$ and $-\infty$ are not numbers.

Theorem 9

Important limits to remember:

- If $a>1$ then $\lim_{n\to\infty} a^n = \infty$
- If $|a|<1$ then $\lim_{n\to\infty} a^n = 0$
- If $a>0$ then $\lim_{n\to\infty} \sqrt[n]{a} = 1$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

Theorem 10 (Properties of infinite limits)

- If $\lim_{n\to\infty} a_n = \pm\infty$ then $\lim_{n\to\infty} \frac{1}{a_n} = 0$ (*vulgar and misleading form* $\frac{1}{\infty} = 0$)
- If $\lim_{n\to\infty} a_n = \infty$ and (b_n) is bounded from below then

  $\lim_{n\to\infty} (a_n + b_n) = \infty$ (*vm form* $\infty + c = \infty$)
- If $\lim_{n\to\infty} a_n = \infty$ and for every n $b_n \geq c$ for some $c>0$, then $\lim_{n\to\infty} a_n b_n = \infty$ (*vm form* $c\infty = \infty$)
- If $\lim_{n\to\infty} a_n = \infty$ and $a_n \leq b_n$ for every n then $\lim_{n\to\infty} b_n = \infty$ (*vm form* squeeze lemma for infinities)